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Extended Binet's formula for the class of Bifurcating Fibonacci sequences-$\{F_n^L\}$ & $\{F_n^R\}$

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Abstract: Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2};\ n \geq 2$ with initial condition $F_0 = 0, F_1 = 1$. This sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation. Two of the generalizations of the Fibonacci sequence are the class of sequences $\left\{F_n^{L(a,b)}\right\}_{n=0}^{\infty}$ and $\left\{F_n^{R(a,b)}\right\}_{n=0}^{\infty}$ generated by the recurrence relation

$$F_n^{L(a,b)} = a^{\chi(n)} b^{1-\chi(n)} F_{n-1}^{L(a,b)} + F_{n-2}^{L(a,b)}$$

and

$$F_n^{R(a,b)} = F_{n-1}^{R(a,b)} + a^{\chi(n)} b^{1-\chi(n)} F_{n-2}^{R(a,b)}$$

with initial condition $F_0^{L(a,b)} = F_0^{R(a,b)} = 0,\ F_1^{L(a,b)} = F_1^{R(a,b)} = 1$ and where a, b are any fixed positive integers and $\chi(n) = \begin{cases} 1; \text{if n is odd} \\ 0; \text{if n is even} \end{cases}$.

The Fibonacci sequence is a special case of these sequences with $a = b = 1$.
In this paper we obtain extended Binet's formula for both $\left\{F_n^{L(a,b)}\right\}_{n=0}^{\infty}$ and $\left\{F_n^{R(a,b)}\right\}_{n=0}^{\infty}$.

Keywords**:** Fibonacci sequence, generalized Fibonacci sequence, Bifurcating Fibonacci sequence, Binet formula.

Introduction

The well-known sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers is a sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … , where each term is the sum of two preceding terms. The corresponding recurrence relation for the terms of the sequence is $F_n = F_{n-1} + F_{n-2}$; $n \geq 2$ with the initial terms $F_0 = 0, F_1 = 1$. There are fundamentally two ways in which the Fibonacci sequence may be generalized; namely, either by maintaining the recurrence relation but altering the first two terms of the sequence from 0, 1 to any arbitrary integers *a*, *b* or by preserving the first two terms of the sequence but altering the recurrence relation. The two techniques can be even combined. It is observed that the change in the recurrence relation lead to greater complexity in the properties of the resulting sequence. Edson, Yayenie(2009) generalized this sequence in to new class of generalized sequence which depends on two real parameters used in a recurrence relation. We define further generalizations of this sequence and call it the generalized Fibonacci sequence.

Definition: For any two positive numbers *a* and *b*, the generalizations of the Fibonacci sequence are the class of sequences $\left\{F_n^{L(a,\,b)}\right\}_{n=0}^{\infty} = \{F_n^L\}$ and $\left\{F_n^{R(a,\,b)}\right\}_{n=0}^{\infty} = \{F_n^R\}$ generated by the recurrence relation

$$F_n^{L(a,b)} = a^{\chi(n)} b^{1-\chi(n)} F_{n-1}^{L(a,b)} + F_{n-2}^{L(a,b)}$$

and

$$F_n^{R(a,b)} = F_{n-1}^{R(a,b)} + a^{\chi(n)} b^{1-\chi(n)} F_{n-2}^{R(a,b)}$$

with initial condition $F_0^{L(a,\,b)} = F_0^{R(a,\,b)} = 0,\ F_1^{L(a,\,b)} = F_1^{R(a,\,b)} = 1$ and where a, b are any fixed positive integers and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

The Fibonacci sequence is a special case of these sequences with $a = b = 1$.

In this paper we derive extended Binet's formula for both $\left\{F_n^{L(a,\,b)}\right\}_{n=0}^{\infty}$ and $\left\{F_n^{R(a,\,b)}\right\}_{n=0}^{\infty}$.

Extended Binet's formula for $\left\{\mathbf{F_n^{L(a,\,b)}}\right\}$:

We first note down a result which can be proved easily.

Lemma 2.1:

For any positive integer *n*, the following holds:

$$\mathrm{F}_{2n+5}^{\mathrm{L(a,\,b)}} - (ab+2)F_{2n+2}^{\mathrm{L(a,\,b)}} + b\mathrm{F}_{2n}^{\mathrm{L(a,\,b)}} = 0.$$

We now obtain the value of two series related with $F_n^{L(a,\,b)}$.

Lemma 2.2:

$$\boldsymbol{\sum_{i=1}^{\infty} \mathbf{F}_{2i-1}^{\mathbf{L(a,\,b)}} x^{2i-1} = \frac{x - x^3}{x^4 - (ab+2)x^2 + 1}}.$$

Proof: Let $p(x) = \sum_{i=1}^{\infty} \mathrm{F}_{2i}^{\mathrm{L(a,\,b)}} x^{2i} = F_2^{\mathrm{L(a,\,b)}} x^2 + \mathrm{F}_4^{\mathrm{L(a,\,b)}} x^4 + \mathrm{F}_6^{\mathrm{L(a,\,b)}} x^6 + \cdots$.

Using Lemma 2.1, we get $(x^4 - (ab+2)x^2 + 1)p(x) = x - x^3$. This gives

$$p(x) = \sum_{i=1}^{\infty} \mathrm{F}_{2i-1}^{\mathrm{L(a,\,b)}} x^{2i-1} = \frac{x - x^3}{x^4 - (ab+2)x^2 + 1}.$$

We next derive the generating function for $\left\{F_n^{L(a,\,b)}\right\}$.

Lemma 2.3:

The generating function for $\left\{F_n^{L(a,\,b)}\right\}$ is given by $f(x) = \frac{x+bx^2-x^3}{x^4-(ab+2)x^2+1}$.

Proof: Define

$$f(x) = \sum_{i=0}^{\infty} \mathrm{F}_{\mathrm{i}}^{\mathrm{L(a,\,b)}} x^i = \mathrm{F}_0^{\mathrm{L(a,\,b)}} + \mathrm{F}_1^{\mathrm{L(a,\,b)}} x + \mathrm{F}_2^{\mathrm{L(a,\,b)}} x^2 + \mathrm{F}_3^{\mathrm{L(a,\,b)}} x^3 + \cdots. \quad (1)$$

After the rearrangement of terms, we get $(1-ax-x^2)f(x) = x + (b-a)x\left(\sum_{i=1}^{\infty} \mathrm{F}_{2\mathrm{i}-1}^{\mathrm{L(a,\,b)}} x^{2i-1}\right)$.

Using Lemma 2.2 and on simplification we get the generating function for $\left\{F_{\mathrm{n}}^{\mathrm{L(a,\,b)}}\right\}$ as $f(x) = \frac{x+bx^2-x^3}{x^4-(ab+2)x^2+1}$.

We are now all set to derive the extended Binet's formula for $F_{\mathrm{n}}^{\mathrm{L(a,\,b)}}$.

Theorem 2.4:

The terms of the generalized Fibonacci sequence $\left\{\mathbf{F}_{\mathbf{n}}^{\mathbf{L(a,\,b)}}\right\}$ are given by

$$\boldsymbol{F_n^{L(a,\,b)} = \sum_{n=0}^{\infty} b^{1-\chi(n)} \left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right)},$$

where $\alpha = \left(\frac{(ab+2)+\sqrt{(ab+2)^2-4}}{2}\right)$, $\beta = \left(\frac{(ab+2)-\sqrt{(ab+2)^2-4}}{2}\right)$ with $\gamma = \alpha - 1$, $\bar{\gamma} = \beta - 1$ and $\chi(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}$.

Proof: Let α and β be the roots of equation $x^2 - (ab+2)x + 1 = 0$.
This gives $\alpha = \left(\frac{(ab+2)+\sqrt{(ab+2)^2-4}}{2}\right)$, $\beta = \left(\frac{(ab+2)-\sqrt{(ab+2)^2-4}}{2}\right)$ so that
$\alpha + \beta = (ab+2), \alpha\beta = 1$.

Now by Lemma 2.4, we have $f(x) = \frac{x+bx^2-x^3}{x^4-(ab+2)x^2+1}$.

If we write $f(x) = \frac{x+bx^2-x^3}{x^4-(ab+2)x^2+1} = \frac{(Ax+B)}{(x^2-\alpha)} + \frac{(cx+D)}{(x^2-\beta)}$ then by using the method of partial fractions, we get

$$f(x) = \frac{1}{(\alpha-\beta)}\left[\frac{(1-\alpha)x+\alpha b}{(x^2-\alpha)} + \frac{(\beta-1)x-b\beta}{(x^2-\beta)}\right]. \quad (2)$$

But it is known that the Maclaurin's expansion of $\frac{P-Qx}{x^2-R}$ is given by

$$\frac{P-Qx}{x^2-R} = \sum_{n=0}^{\infty} QR^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} PR^{-n-1}x^{2n}.$$

Then $\frac{\alpha b-(\alpha-1)x}{(x^2-\alpha)} = \left[\sum_{n=0}^{\infty} \frac{(\alpha-1)}{(\alpha)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{\alpha b}{\alpha^{n+1}} x^{2n}\right]$ and
$\frac{(-b\beta-(1-\beta)x)}{(x^2-\beta)} = \left[\sum_{n=0}^{\infty} \frac{(1-\beta}{(\beta)^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{-b\beta}{(\beta)^{n+1}} x^{2n}\right]$.

Using these in (2) we get

$$f(x) = \frac{1}{(\alpha-\beta)}\left[\left\{\sum_{n=0}^{\infty}\left(\frac{\alpha-1}{\alpha^{n+1}} - \frac{\beta-1}{\beta^{n+1}}\right)x^{2n+1} - \sum_{n=0}^{\infty}\left(\frac{b\alpha}{\alpha^{n+1}} - \frac{b\beta}{\beta^{n+1}}\right)x^{2n}\right\}\right].$$

Since $\alpha\beta = b$, we get

$$f(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left((1-\beta)\alpha^{n+1} - (1-\alpha)\beta^{n+1}\right)x^{2n+1} + \sum_{n=0}^{\infty} b(\alpha^n - \beta^n)x^{2n}\right\}.$$

But $(1-\beta)\alpha^{n+1} = (\alpha-1)\alpha^n$ and $(1-\alpha)\beta^{n+1} = (\beta-1)\beta^n$. For brevity we write $\gamma = \alpha - 1$ and $\bar{\gamma} = \beta - 1$. Thus we have

$$f(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}(\gamma\alpha^n - \bar{\gamma}\beta^n)x^{2n+1} - \sum_{n=0}^{\infty} b(\alpha^n - \beta^n)x^{2n}\right\}.$$

On defining $\chi(n) = 1$, if *n* is odd and $\chi(n) = 0$, if *n* is even we write $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right)x^n. \qquad (3)$$

Finally by (2) and (3) we have

$$\mathrm{F}_{\mathrm{n}}^{\mathrm{L(a,\,b)}} = \sum_{n=0}^{\infty} b^{1-\chi(n)}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right).$$

Extended Binet's formula for $\left\{\mathbf{F}_{\mathbf{n}}^{\mathbf{R(a,\,b)}}\right\}$

In this section we use the techniques similar to that used in the above section.

Lemma 3.1:

For any positive integer *n*, the following holds:

$$\mathrm{F}_{2\mathrm{n}+4}^{\mathrm{R(a,\,b)}} - (a+b+1)\mathrm{F}_{2\mathrm{n}+2}^{\mathrm{R(a,\,b)}} + ab\mathrm{F}_{2\mathrm{n}}^{\mathrm{R(a,\,b)}} = 0\,.$$

Here too we obtain the value of two series related with $\mathrm{F}_{\mathrm{n}}^{\mathrm{R(a,\,b)}}$.

Lemma 3.2:

$$\boldsymbol{\sum_{i=0}^{\infty} \mathrm{F}_{2\mathrm{i}}^{\mathrm{R(a,\,b)}} x^{2i} = \frac{x^2}{abx^4 - (a+b+1)x^2 + 1}}\,.$$

Proof: Let $u(x) = \sum_{i=0}^{\infty} \mathrm{F}_{2\mathrm{i}}^{\mathrm{R(a,\,b)}} x^{2i} = \mathrm{F}_{0}^{\mathrm{R(a,\,b)}} + F_{2}^{\mathrm{R(a,\,b)}} x^2 + \mathrm{F}_{4}^{\mathrm{R(a,\,b)}} x^4 + \mathrm{F}_{6}^{\mathrm{R(a,\,b)}} x^6 + \cdots$.

Using Lemma 3.1, we get $(abx^4 - (a+b+1)x^2 + 1)u(x) = x^2$, which gives

$$u(x) = \sum_{i=0}^{\infty} \mathrm{F}_{2\mathrm{i}}^{\mathrm{R(a,\,b)}} x^{2i} = \frac{x^2}{abx^4 - (a+b+1)x^2 + 1}\,.$$

We now derive the generating function for $\left\{\mathrm{F}_{\mathrm{n}}^{\mathrm{R(a,\,b)}}\right\}$.

Lemma 3.3:

The generating function for $\left\{\boldsymbol{F_n^{R(a,\,b)}}\right\}$ is given by

$$\boldsymbol{l(x) = \frac{x^2}{abx^4 - (a+b+1)x^2 + 1}}\,.$$

Proof: Define

$$l(x) = \sum_{i=0}^{\infty} \mathrm{F}_{\mathrm{i}}^{\mathrm{R(a,\,b)}} x^i = \mathrm{F}_{0}^{\mathrm{R(a,\,b)}} + \mathrm{F}_{1}^{\mathrm{R(a,\,b)}} x + \mathrm{F}_{2}^{\mathrm{R(a,\,b)}} x^2 + \mathrm{F}_{3}^{\mathrm{R(a,\,b)}} x^3 + \cdots. \qquad (4)$$

After the rearrangement of terms, we get

$$(1 - x - ax^2)l(x) = x + x^2(b - a)\left(\sum_{i=0}^{\infty} \mathrm{F}_{2\mathrm{i}}^{\mathrm{R(a,\,b)}} x^{2i}\right)$$

Using Lemma 3.2 and on simplification we get the generating function for $\left\{\mathrm{F}_{\mathrm{n}}^{\mathrm{R(a,\,b)}}\right\}$ as $l(x) = \frac{x^2}{abx^4 - (a+b+1)x^2 + 1}$.

We finally obtain the extended Binet's formula for $\mathrm{F}_{\mathrm{n}}^{\mathrm{R(a,\,b)}}$.

Theorem 3.4:

The terms of the generalized Fibonacci sequence $\left\{\boldsymbol{F_n^{R(a,\,b)}}\right\}$ are given by

$$\boldsymbol{F_n^{R(a,\,b)} = \sum_{n=0}^{\infty}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha - \beta}\right)},$$

where $\boldsymbol{\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4ab}}{2}\right)}$, $\boldsymbol{\beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4ab}}{2}\right)}$ with $\boldsymbol{\gamma = \alpha - b}$, $\boldsymbol{\bar{\gamma} = \beta - b}$ and $\boldsymbol{(n) = \begin{cases} 1; \text{if } n \text{ is odd} \\ 0; \text{if } n \text{ is even} \end{cases}}$.

Proof: Let α and β be the roots of equation $x^2 - (a + b + 1)x + ab = 0$.

This gives $\alpha = \left(\frac{(a+b+1)+\sqrt{(a+b+1)^2-4ab}}{2}\right), \beta = \left(\frac{(a+b+1)-\sqrt{(a+b+1)^2-4ab}}{2}\right)$. Also we have $\alpha + \beta = (a + b + 1), \alpha\beta = ab$.

Using Lemma 3.3, we have $l(x) = \frac{abx + abx^2 - ab^2x^3}{abx^4 - (a+b+1)x^2 + 1}$.

If we write $l(x) = \frac{abx + abx^2 - ab^2x^3}{a^2b^2x^4 - ab(a+b+1)x^2 + ab} = \frac{(Ax+B)}{(abx^2-\alpha)} + \frac{(cx+D)}{(abx^2-\beta)}$ then by using the method of partial fractions, we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left[\frac{(-\alpha b + ab)x + \alpha}{(abx^2-\alpha)} + \frac{(b\beta - ab)x - \beta}{(abx^2-\beta)}\right]. \tag{5}$$

Now by using Maclaurin's expansion we get

$$\frac{\alpha - (\alpha b - ab)x}{ab\left(x^2 - \alpha/_{ab}\right)} = \frac{1}{ab}\left[\sum_{n=0}^{\infty}\frac{\alpha b - ab}{\left(\alpha/_{ab}\right)^{n+1}}x^{2n+1} - \sum_{n=0}^{\infty}\frac{\alpha}{\left(\alpha/_{ab}\right)^{n+1}}x^{2n}\right] \text{ and}$$

$$\frac{-\beta - (b\beta - ab)x}{ab\left(x^2 - \beta/_{ab}\right)} = \frac{1}{ab}\left[\sum_{n=0}^{\infty}\frac{-(b\beta - ab)}{\left(\beta/_{ab}\right)^{n+1}}x^{2n+1} + \sum_{n=0}^{\infty}\frac{\beta}{\left(\beta/_{ab}\right)^{n+1}}x^{2n}\right].$$

Using these in (5) we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left[\frac{(ab)^{n+1}}{ab}\left\{\sum_{n=0}^{\infty}\left(\frac{\alpha b - ab}{\alpha^{n+1}} + \frac{ab - b\beta}{\beta^{n+1}}\right)x^{2n+1} - \sum_{n=0}^{\infty}\left(\frac{\alpha}{\alpha^{n+1}} - \frac{\beta}{\beta^{n+1}}\right)x^{2n}\right\}\right].$$

Using $\alpha\beta = ab$, we get

$$l(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}\left((\alpha - b)\alpha^n - (\beta - b)\beta^n\right)x^{2n+1} + \sum_{n=0}^{\infty}(\alpha^n - \beta^n)x^{2n}\right\}.$$

Writing $\gamma = \alpha - b$ and $\bar{\gamma} = \beta - b$, we have

$$l(x) = \frac{1}{(\alpha-\beta)}\left\{\sum_{n=0}^{\infty}(\gamma\alpha^n - \bar{\gamma}\beta^n)x^{2n+1} + \sum_{n=0}^{\infty}(\alpha^n - \beta^n)x^{2n}\right\}.$$

$$= \sum_{n=0}^{\infty}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right)x^n.$$

This gives

$$F_n^{R(a,\,b)} = \sum_{n=0}^{\infty}\left(\frac{\gamma^{\chi(n)}\alpha^{[n/2]} - \bar{\gamma}^{\chi(n)}\beta^{[n/2]}}{\alpha-\beta}\right).$$

Conclusions

In this paper, we considered the sequence of 'Bifurcating Fibonacci numbers' and obtained extended Binet's formula for both $\left\{F_n^{L(a,\,b)}\right\}_{\mathrm{n}=0}^{\infty}$ and $\left\{F_n^{R(a,\,b)}\right\}_{n=0}^{\infty}$.

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