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Various properties related with the class of second order linear homogeneous recurrence relations with constant coefficients

Dr. Vandana R. Patel, Department of Mathematics, Sir P.T.Science College, Modasa, India
Dr. Devbhadra V. Shah, Department of Mathematics, Veer Narmad South Gujarat University, Surat, India
Parimal A. Patel, Department of Mathematics, C. B. Patel Computer College & J. N. M. Patel Science College, Bharthana, Vesu, Surat, Gujarat, India
Dr. Daksha M. Diwan, Department of Mathematics, Government Engineering College, Gandhinagar, India

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Abstract: In the present work we consider the class of second order linear homogeneous recurrence relation depending on two parameters which has the form $x_n = ax_{n-1} + bx_{n-2}\,; n \geq 2$ with initial condition $x_0 = 0, x_1 = 1$and a, b are positive integers. If $a = b = 1$, $\{x_n\}$ is sequence of Fibonacci numbers. In the case $a = 2, b = 1$,$\{x_n\}$ is a sequence of Pell numbers. In this paper we obtain Binet type formula for $\{x_n\}$and express $x_n$in simple explicit form. We use it to derive the recursive formulafor x_n and also compute its successor and predecessor. We also find the bounds and various results for the powers of corresponding 'golden proportion' for this sequence.

Keywords: Linear recurrence relation, Fibonacci sequence, Generalized Fibonacci sequence, Golden proportion.

AMS subject classification number: 11B39, 11B37.

Introduction

Fibonacci numbers are perhaps most famous for appearing in the rabbit breeding problem, introduced by Leonardo de Pisa in 1202 in his book called Liber Abaci, they remain fascinating and mysterious to people today. However, they also occur in Pascal's triangle[Koshy],in Pythagorean triples[Koshy],computer algorithms[Stojmenovic;Fredman,Tarjan],some areas of algebra[Feingold;Suck,Schreiber,P.Häussler;Schork],graphTheory[Chebotarev;Bogdonowicz,],quasi - crystals[Atkins,Geist; Zubov,Teixeira Rabelo],and many areas of mathematics. They occur in a variety of other fields such as finance, art, architecture, music, etc. The Fibonacci sequence is a source of many identities as appears in the work of Vajda , Harris, and Carlitz.

The Fibonacci sequence $\{F_n\}$ is defined by $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. also the sequence of Lucas numbers $\{L_n\}$ is defined by $L_n = L_{n-1} + L_{n-2}$, for all $n \geq 2$, with initial conditions $L_0 = 2$ and $L_1 = 1$.

The Binet's formula for Fibonacci sequence and Lucas sequence is given by $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ and $L_n = \alpha^n + \beta^n = \left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ respectively.

Where $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ =Golden ratio $= 1.618$ and $\beta = \left(\frac{1-\sqrt{5}}{2}\right) = -0.618$

In this paper, we present different properties of the Generalized Fibonacci sequence $\{G_n\}$ which is defined by $G_n = aG_{n-1} + bG_{n-2}$, for all $n \geq 2$ with $G_0 = 0$ and $G_1 = 1$; where a and b are nonzero real numbers.

The few terms of the sequence $\{G_n\}$ are : 0,1,a,a2+2ab,a4+3a2b+b2,a5+4a3b+3ab2,…and so on. The Generalized Lucas sequence $\{K_n\}$ which is defined by $K_n = aK_{n-1} + bK_{n-2}$, for all $n \geq 2$ with $K_0 = 2$ and $K_1 = 1$; where a and b are nonzero real numbers.

Generating Function For The Generalized Fibonacci Sequence

Generating functions provide a powerful method for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with nonconstant coefficients. In this section, we consider the generating function for the generalized Fibonacci sequence and derive some of the most interesting identities satisfied by this sequence.

Theorem 2.1. The generating function for the generalized Fibonacci sequence given by $\{G_n\}$ is $g(x) = \frac{x}{1-ax-bx^2}$.

Proof: Let $g(x) = G_0 + G_1x + G_2x^2 + \cdots + G_nx^n + \cdots$ be the generating function of the generalized Fibonacci sequence $\{G_n\}$, we note that $G_0 = 0$, $G_1 = 1$.

Now, $g(x) = G_0 + G_1x + G_2x^2 + \cdots + G_nx^n + \cdots$

$$axg(x) = aG_0x + aG_1x^2 + aG_2x^3 + \cdots + aG_nx^{n+1} + \cdots$$

$$bx^2g(x) = bG_0x^2 + bG_1x^3 + bG_2x^4 + \cdots + bG_nx^{n+2} + \cdots$$

We will add the power series $g(x)$, $-axg(x)$, $-bx^2g(x)$ then we get,

$$g(x) - axg(x) - bx^2g(x) = G_0 + (-aG_0 + G_1)x + (-bG_0 - aG_1 + G_2)x^2 + \ldots\ldots\ldots$$

Here notice that if we take our rearranged recursion formula $G_n - aG_{n-1} + bG_{n-2} = 0$,with $n = 2$, we get $G_2 - aG_1 + bG_0 = 0$. Thus, the Co efficient of x^2 term in our combined series is zero. In fact using the recursion formula, the co efficient of the terms after the x^2 term we see they are all zero.

Thus We have $g(x) - axg(x) - bx^2g(x) = G_0 + (-aG_0 + G_1)x$, Since $G_0 = 0$, $G_1 = 1$

$\therefore (1 - ax - bx^2)g(x) = x$

$\therefore g(x) = \frac{x}{1-ax-bx^2} = \sum_{n=0}^{\infty} G_nx^n$, which is required generating function.

Binet's Formula For The Generalized Fibonacci Sequence

Koshy refers to the Fibonacci numbers as one of the "two shining stars in the vast array of integer sequences "[Koshy]. We may guess that one reason for this reference is the sheer quantity of interesting properties this sequence possesses. Further still, almost all of these properties can be derived from Binet's formula. A main objective of this paper is to demonstrate that many of the properties of the Fibonacci sequence can be stated and proven for a much larger class of sequences, namely the generalized Fibonacci sequence. Therefore, we will state and prove Binet's formula for the generalized Fibonacci sequence.

Theorem 3.1: (Binet's Formula)) The terms of the generalized Fibonacci sequence $\{G_n\}$ are given by $G_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$; where $\alpha = \frac{a+\sqrt{a^2+4b}}{2}$ and $\beta = \frac{a-\sqrt{a^2+4b}}{2}$ (3.1)

Proof: We first express function $g(x)$ for G_n as a sum of partial fractions.
Let $(1 - ax - bx^2) = (1 - \alpha x)(1 - \beta x)$
Now Consider $g(x) = \frac{x}{1-ax-bx^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$
$\therefore x = A(1 - \beta x) + B(1 - \alpha x)$
If $x = \frac{1}{\beta}$ then $\frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right) \Rightarrow \frac{1}{\beta} = B\left(\frac{\beta-\alpha}{\beta}\right)$

$$\Rightarrow B = \frac{1}{\beta - \alpha} \quad \Rightarrow B = \frac{-1}{\alpha - \beta}$$

Similarly, If We take $x = \frac{1}{\alpha}$ then we have $\frac{1}{\alpha} = A(1 - \frac{\beta}{\alpha}) \Rightarrow A = \frac{1}{\alpha-\beta}$

$$\therefore g(x) = \frac{x}{1 - ax - bx^2} = \frac{\frac{1}{\alpha - \beta}}{1 - \alpha x} + \frac{\frac{-1}{\alpha - \beta}}{1 - \beta x}$$

$\therefore g(x) = \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty} \alpha^n x^n - \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty} \beta^n x^n$
$\therefore g(x) = \sum_{n=0}^{\infty} \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) x^n$
But , $g(x) = \sum_{n=0}^{\infty} G_n\, x^n$
$\therefore G_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ Which is Binet's formula for given generalized Fibonacci sequence.

Lemma: 3.2. $|\beta| < 1$; when $b \leq a$.

Theorem 3.3.: $G_n = \left[u\alpha^n + \frac{1}{2}\right]; for\ n \geq 2$, where $u = \frac{1}{\sqrt{a^2+4b}}$ and $a \geq b,\ a > 0\ and\ b > 0$ are integers

Proof: By theorem 3.1, we have,

$$|G_n - u\alpha^n| = \left|\frac{\alpha^n}{\sqrt{a^2+4b}} - \frac{\beta^n}{\sqrt{a^2+4b}} - \frac{\alpha^n}{\sqrt{a^2+4b}}\right|$$
$$= \left|-\frac{\beta^n}{\sqrt{a^2+4b}}\right|$$
$$= \frac{|\beta|^n}{\sqrt{a^2+4b}}$$

Using Lemma 3.2, we get $|G_n - u\alpha^n| < \frac{1}{\sqrt{a^2+4b}}$
But $\sqrt{a^2 + 4b} > 2$ as a and b are positive integers
$\Longrightarrow \frac{1}{\sqrt{a^2+4b}} < \frac{1}{2}$
$\therefore |G_n - u\alpha^n| < \frac{1}{2}$
Thus for $n \geq 2$, $G_n = \left[u\alpha^n + \frac{1}{2}\right]$, as required.
For example, let $a = 4$, $b = 3$, $G_0 = 0$ and $G_1 = 1$ then $\alpha = 4.646$ and $u = 0.18896$
$\alpha^5 = 2164.226$ so, $\alpha^5 u = 408.95$

$\therefore G_5 = \left[408.95 + \frac{1}{2}\right]$ = [409.45] = 409

Also, by calculating, $G_5 = 409$

Theorem 3.4. : $G_{n+1} = [\alpha G_n + 1]$; for $n \geq 2$

Proof: by theorem 3.1 we have,

$$G_{n+1} = \frac{\alpha^{n+1}}{\sqrt{a^2+4b}} - \frac{\beta^{n+1}}{\sqrt{a^2+4b}} \text{ and } \alpha G_n = \frac{\alpha^{n+1}}{\sqrt{a^2+4b}} - \frac{\alpha\beta^n}{\sqrt{a^2+4b}}$$

Substracting these two results, we get

$$G_{n+1} - \alpha G_n = \frac{\alpha\beta^n}{\sqrt{a^2+4b}} - \frac{\beta^{n+1}}{\sqrt{a^2+4b}} = \frac{\beta^n(\alpha-\beta)}{\sqrt{a^2+4b}} = \frac{\beta^n(\sqrt{a^2+4b})}{\sqrt{a^2+4b}} = \beta^n$$

Using Lemma 3.2 , for $n \geq 2$

We get, $G_{n+1} - \alpha G_n < 1$

$$\therefore G_{n+1} < \alpha G_n + 1 \quad (3.2)$$

On the other hand again by lemma 3.2 , $G_{n+1} - \alpha G_n > -1$, for $n \geq 2$

$$\alpha G_n + 1 < G_{n+1} + 2 \quad (3.3)$$

Combining (3.2) and (3.3), we get $G_{n+1} < \alpha G_n + 1 < G_{n+1} + 2$

$$\Longrightarrow G_{n+1} < \alpha G_n + 1 \leq G_{n+1} + 1$$

For example, let $a = 3$, $b = 2$ then $\alpha = 3.5615$; $G_5 = [\alpha G_4 + 1] = [138.89 + 1] = [139.89] = 139$

Also by calculating $G_5 = 139$

Corollary 3.5. : $\frac{G_{n+1}}{G_n} = \alpha$.

Theorem 3.6 : $G_n = \left[\frac{1}{\alpha}(G_{n+1} + 1)\right]$; for $n \geq 2$

Proof: since $x - 1 < [x] \leq x$, theorem 3.4 gives the double inequality

$$\alpha G_n - 1 < G_{n+1} < \alpha G_n + 1$$

$$G_n - \frac{1}{\alpha} < G_{n+1} < G_n + \frac{1}{\alpha} \quad \text{as } \alpha > 0$$

Then $G_n \leq \frac{1}{\alpha}(G_{n+1} + 1)$ and $G_n > \frac{1}{\alpha}(G_{n+1} - 1)$

$$\therefore \frac{1}{\alpha}(G_{n+1} - 1) < G_n \leq \frac{1}{\alpha}(G_{n+1} + 1)$$

Since $\frac{1}{\alpha}(G_{n+1} + 1) - \frac{1}{\alpha}(G_{n+1} - 1) = \frac{2}{\alpha} < 1$ and G_n is an integer, it follows that

$$G_n = \left[\frac{1}{\alpha}(G_{n+1} + 1)\right] \text{; for } n \geq 2$$

For example, the predecessor of the Generalized Fibonacci number $G_n = 1189$ is given by

$\left[\frac{G_7+1}{\alpha}\right] = \left[\frac{1190}{3.3028}\right] = [360.30] = 360$; which is sixth Generalized Fibonacci number. Where $a = 3$, $b = 1$ then $\alpha = 3.302775 \approx 3.3028$

Before concluding this section, we prove the result which gives nice bounds for G_n; for $n \geq 3$.

Theorem 3.7 : $\alpha^{n-2} < G_n < \alpha^{n-1}$; for $n \geq 3$.

Proof: we prove the result by induction on n.

For $n = 3$, we have to prove $\alpha < G_3 < \alpha^2$
We know that $\alpha^2 - a\alpha - b = 0 \Longrightarrow \alpha^2 = a\alpha + b$
So, we have to only show that $\alpha < G_3 < a\alpha + b$
Now, from theorem 3.4, $G_3 = [\alpha G_2 + 1] = \alpha G_2 + 1 + \theta$; where $0 \leq \theta < 1$
$G_3 = a\alpha + 1 + \theta$ as $G_2 = a$
Thus, clearly $G_3 > \alpha$ as $\alpha > 1$ and $\theta \geq 0$ (3.4)
Here first we claim that $\alpha > a$
We have $\alpha - a = \frac{a+\sqrt{a^2+4b}}{2} - a = \frac{\sqrt{a^2+4b}-a}{2} = \frac{a}{2}\left(\sqrt{1+\frac{4b}{a^2}} - 1\right) > 0$
$\therefore \alpha - a > 0 \Longrightarrow \alpha > a$
Also, $G_3 = aG_2 + bG_1 = a.a + b < \alpha.a + b$ as $\alpha > a$
$\therefore G_3 < a\alpha + b$ (3.5)
Combining (3.4) and (3.5), we get $\alpha < G_3 < a\alpha + b \Longrightarrow \alpha < G_3 < \alpha^2$
So, the result is true for $n = 3$.
Assume that result is true for all n, such that $3 \leq n \leq m$, that is $\alpha^{m-2} < G_m < \alpha^{m-1}$ and $\alpha^{m-3} < G_{m-1} < \alpha^{m-2}$ holds.
Multiplying first inequality by 'a' and second by 'b' then adding them, we get
$a\alpha^{m-2} + b\alpha^{m-3} < aG_m + bG_{m-1} < a\alpha^{m-1} + b\alpha^{m-2}$
$\therefore \alpha^{m-3}(a\alpha + b) < G_{m+1} < \alpha^{m-2}(a\alpha + b)$
$\therefore \alpha^{m-3}\alpha^2 < G_{m+1} < \alpha^{m-2}\alpha^2$ as $\alpha^2 = a\alpha + b$
$\therefore \alpha^{m-1} < G_{m+1} < \alpha^m$
$\therefore \alpha^{(m+1)-2} < G_{m+1} < \alpha^{(m+1)-1}$
This proves the result by induction, for $n \geq 3$.

Value of α^n :

It is well known that if α is real root of the equation $x^2 - x - 1 = 0$ then $\alpha^n = \frac{1}{2}(L_n \pm \sqrt{5}F_n)$, where F_n and L_n respectively are the nth Fibonacci and Lucas numbers. Here we establish analogous identity for the Generalized Fibonacci numbers.

Theorem 4.1 : $\alpha^n = \frac{1}{2}(L_n + \sqrt{a^2+4b}G_n)$, $n \geq 1$; where recurrence relation for L_n is given as : $L_n = aL_{n-1} + bL_{n-2}$ with initial conditions $L_0 = 2, L_1 = a$ and G_n is a Generalized Fibonacci number.

Proof: we prove the result by induction on n.
For $n = 1$, we have $\frac{1}{2}\left(L_1 + \sqrt{a^2+4b}G_1\right) = \frac{1}{2}\left(a + \sqrt{a^2+4b}\right) = \alpha$
Thus, the result is true for $n = 1$.
Suppose result is true for $n = m$; so, $\alpha^m = \frac{1}{2}(L_m + \sqrt{a^2+4b}G_m)$ holds
Now, $\frac{1}{2}\left(L_{m+1} + \sqrt{a^2+4b}G_{m+1}\right) = \frac{1}{2}(aL_m + bL_{m-1} + \sqrt{a^2+4b}(aG_m + bG_{m-1}))$

$$= a\left(\frac{L_m+\sqrt{a^2+4b}G_m}{2}\right)+b\left(\frac{L_{m-1}+\sqrt{a^2+4b}G_{m-1}}{2}\right)$$
$$= a\alpha^m + b\,\alpha^{m-1}$$
$$= \alpha^{m-1}(a\alpha+b)$$
$$= \alpha^{m-1}\,\alpha^2 \quad \text{as} \quad \alpha^2 = a\alpha+b$$
$$= \alpha^{m+1}$$

Therefore, we get $\alpha^{m+1} = \frac{1}{2}\left(L_{m+1}+\sqrt{a^2+4b}G_{m+1}\right)$

Thus, the result is true for $n = m+1$ also.

Hence, the result is true for all $n \geq 1$,this proves the required result.

Theorem 4.2 : $\alpha^n = G_n\alpha + bG_{n-1}$ and $\beta^n = G_n\beta + bG_{n-1}$, $\forall n \in N$.

Proof : we prove this result by induction on n,

if $n = 1$ then $G_1\alpha + bG_0 = \alpha + 0 = \alpha$ as $G_1 = 1\ and\ G_0 = 0$

Therefore result is true for $n = 1$.

Suppose result is true for $n = m$.

$\therefore \alpha^m = G_m\alpha + bG_{m-1}$ holds.

Now we have to show that result is true for $n = m+1$.

$$\therefore\ \alpha^{m+1} = \alpha\alpha^m = \alpha(G_m\alpha + bG_{m-1})$$
$$= \alpha^2G_m + \alpha bG_{m-1}$$
$$= (a\alpha+b)G_m + \alpha bG_{m-1} \quad (\because \alpha^2 = a\alpha+b$$
$$= \alpha(G_m\alpha + bG_{m-1}) + bG_m$$
$$= \alpha G_{m+1} + bG_m$$

Thus result is true for $n = m+1$.

So, by mathematical induction we can say that,

$\alpha^n = G_n\alpha + bG_{n-1}$, for all $n \in N$

Similarly, we can prove $\beta^n = G_n\beta + bG_{n-1}$

Theorem 4.3 : $\alpha^n = \frac{1}{2}(L_n + (2\alpha-1)G_n)$, for all $n \in N$; where $L_n = aL_{n-1} + bL_{n-2}$with initial conditions $L_0 = 2, L_1 = 1$.

Proof : we prove this result by induction on n.

For $n = 1$, $\frac{1}{2}(L_1 + (2\alpha-1)G_1) = \frac{1}{2}(1+2\alpha-1) = \alpha$

$\therefore$ This result is true for $n = 1$.

Suppose given result is true for $n = m$; so, $\alpha^m = \frac{1}{2}(L_m + (2\alpha-1)G_m)$.

Now, $\frac{1}{2}(L_{m+1} + (2\alpha-1)G_{m+1}) = \frac{1}{2}\{aL_m + bL_{m-1} + (2\alpha-1)(aG_m + bG_{m-1})\}$

$$= a\left(\frac{L_m+(2\alpha-1)G_m}{2}\right)+b\left(\frac{L_{m-1}+(2\alpha-1)G_{m-1}}{2}\right)$$
$$= a\,\alpha^m + b\,\alpha^{m-1}$$
$$= \alpha^{m-1}(a\alpha+b)$$
$$= \alpha^{m-1}\alpha^2 \quad \text{as } \alpha^2 = a\alpha+b$$
$$= \alpha^{m+1}$$

$\therefore \alpha^{m+1} = \frac{1}{2}(L_{m+1} + (2\alpha-1)G_{m+1})$

Thus, the result is true for $n = m + 1$ also.

The number of digits in a Generalized Fibonacci number :

Binet type formula (3.1) can be successfully used to determine the number of digits in G_n(we denote it by $\#G_n$). We prove the following result:

Theorem 5.1 : $\#G_n = [n \log \log \alpha - \log \log |\alpha - \beta|] + 1$; where 'log' represents the logarithm with base 10.

Proof : we write (3.1) as $G_n = \frac{\alpha^n}{\alpha-\beta}\left[1 - \left(\frac{\beta}{\alpha}\right)^n\right]$

$\therefore |G_n| = \frac{\alpha^n}{|\alpha-\beta|}\left[1 - \left|\frac{\beta}{\alpha}\right|^n\right]$

Since $|\beta| < |\alpha|$, we have $\left|\frac{\beta}{\alpha}\right|^n \to 0$ as $n \to \infty$

Therefore, when n is sufficiently large, $G_n \approx \frac{\alpha^n}{|\alpha-\beta|}$

$\therefore \log \log G_n \approx n \log \log \alpha - \log \log |\alpha - \beta|$

$\therefore$The number of digits in $G_n = 1 +$characteristic of $\log \log G_n$

$\therefore \#G_n = [\log \log G_n] + 1$

Thus, $\#G_n = [n \log \log \alpha - \log \log |\alpha - \beta|] + 1$

Remark : if $a = 2, b = 3$ then and $|\alpha - \beta| = 4$

$\therefore$ we have number of digits in $G_8 = [8 \log \log 3 - \log \log 4] + 1 = [3.2149] + 1 = 3 + 1 = 4$

Notice that $G_8 = 1640$ does indeed contain 4 digits.

Convergence of Generalized Fibonacci decimal expansion:

In 1953, stancliff observed that$\sum_{i=1}^{\infty} \frac{F_i}{10^{i+1}} = \frac{1}{89}$; where F_i is the ith Fibonacci number. Koshy(p.p.425) proved that $\sum_{i=1}^{\infty} \frac{F_i}{m^{i+1}} = \frac{1}{m^2-m-1}$;where m is any positive integer. Notice that the denominator is the characteristic polynomial for Fibonacci sequence. In this last article, we prove an analogous result for Generalized Fibonacci sequence.

Theorem 6.1: For any positive integer m, $\sum_{i=1}^{\infty} \frac{G_i}{m^{(i+1)n}} = \frac{1}{m^{2n}-am^n-b}$

Proof: by Theorem 2.1, the generating function for Generalized Fibonacci sequence $\{G_n\}$n≥0 is

$$g(x) = \frac{x}{1-ax-bx^2} = \sum_{i=0}^{\infty} G_i x^i \qquad (6.1)$$

Since, $|1 - ax - bx^2| \geq 1 - a|x| - b|x|^2 > 0$

i.e. $b|x|^2 + a|x| - 1 < 0$

so, $|x| = \frac{-a \pm \sqrt{a^2+4b}}{2b}$

also, $a + b > 1$; $4ab + 4b^2 > 4b$; $a^2 + 4ab + 4b^2 > a^2 + 4b \Rightarrow \pm\sqrt{a^2 + 4b} < a + 2b$

$\Rightarrow \frac{-a \pm \sqrt{a^2+4b}}{2b} < 1 \Rightarrow |x| < 1$

Thus, $|1 - ax - bx^2| > 0$ if $|x| < 1$, the power series expansion of $g(x)$ is absolutely convergent for all x with $|x| < 1$. Let $x = m^{-n}$ with $n \geq 1$, thus (6.1) gives :

$\sum_{i=0}^{\infty} G_i m^{-ni} = \frac{m^{-n}}{1-am^{-n}-bm^{-2n}} \Longrightarrow \sum_{i=0}^{\infty} \quad G_i m^{-ni} = \frac{m^n}{m^{2n}-am^n-b}$

Divides both sides by m^n, we get

$\sum_{i=0}^{\infty} G_i m^{-ni-n} = \frac{1}{m^{2n}-am^n-b}$

$\sum_{i=0}^{\infty} \frac{G_i}{m^{(i+1)n}} = \frac{1}{m^{2n}-am^n-b}$, as required result.

Corollary 6.2 : $\sum_{i=0}^{\infty} \frac{G_i}{10^{(i+1)}} = \frac{1}{100-10a-b}$

Proof: If we take $m = 10$, $n = 1$ in theorem(6.1) then we can easily get required result.

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